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A Uniqueness Theorem for H^p Functions*

ROBERT P. FEINERMAN

Harvard University, Cambridge, Massachusetts 02138

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A function $F(z)$, analytic in the unit disc, is said to be of class H^p $(0 < p < \infty)$ if $\int_{-\pi}^{\pi} | F(re^{i\theta}) |^p \, d\theta$ is bounded as $r \to 1$. It is well known (see [2]) that if $F \in H^p$ $(p \geqslant 1)$ then, for almost all $\theta \in [-\pi, \pi]$, $\lim F(z)$ as $z \to e^{i\theta}$ nontangentially, exists. Moreover, calling $F(\theta)$ that limit, we can recover $F(z)$ from $F(\theta)$ by taking the convolution of $F(\theta)$ and the Poisson kernel.

In this paper, we do not start out with the value of $F(\theta)$ but rather with some specified knowledge of $\arg F(\theta)$. In terms of the given conditions we are able to determine $F(z)$ explicitly.

Notation. We always have $p \geqslant 1$, λ real and $\pi\beta = 2 \arctan \lambda$.
We notice that if $G(z) = (1 - z/1 + z)^\beta$ then

(1) $G \in H^p$ for $p \, | \beta | < 1$

(2) $G(z)$ has real Taylor-coefficients about 0

(3) $\lambda \operatorname{Re} G(\theta) = -\operatorname{Im} G(\theta)$ a.e. in $(0, \pi)$.

We shall prove that (2) is really just a condition on $G(\theta)$. What is really curious is that conditions (1), (2) and (3) are enough to determine $G(z)$. That is our

MAIN THEOREM. *If $F(z)$, analytic in the unit disc, is in H^p, has real coefficients and satisfies $\lambda \operatorname{Re} F(\theta) = -\operatorname{Im} F(\theta)$ a.e. in $(0, \pi)$, then*

$$F(z) = C \left(\frac{1 - z}{1 + z}\right)^\beta$$

where C is real and (C is 0 if $p \, | \beta | \geqslant 1$).

LEMMA. *Let $F(z)$ be in H^p. Then $F(z)$ has real coefficients iff $\operatorname{Re} F(\theta) = \operatorname{Re} F(-\theta)$ and $\operatorname{Im} F(\theta) = -\operatorname{Im} F(-\theta)$ a.e. in $(0, \pi)$.*

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PROOF. Assume $F(z)$ has real coefficients.

$$F(z) = \sum_{n=0}^{\infty} a_n z^n \quad a_n \text{ real}$$

$$F(\bar{z}) = \sum_{n=0}^{\infty} a_n (\bar{z})^n = \overline{\left(\sum_{n=0}^{\infty} a_n z^n\right)} = \overline{F(z)}$$

or writing $z = re^{i\theta}$ we have

$$\operatorname{Re} F(re^{-i\theta}) + i \operatorname{Im} F(re^{-i\theta}) = \operatorname{Re} F(re^{i\theta}) - i \operatorname{Im} F(re^{i\theta})$$

or

$$\operatorname{Re} F(re^{-i\theta}) = \operatorname{Re} F(re^{i\theta}) \qquad \text{and} \qquad \operatorname{Im} F(re^{i\theta}) = -\operatorname{Im} F(re^{-i\theta}).$$

Letting $r \to 1$ we get the desired equalities.

Now assume $\operatorname{Re} F(-\theta) = \operatorname{Re} F(\theta)$ and $\operatorname{Im} F(-\theta) = -\operatorname{Im} F(\theta)$ a.e. in $(0, \pi)$.

$$2\pi a_n = \int_{-\pi}^{\pi} F(\theta)\, e^{in\theta}\, d\theta = \int_{-\pi}^{\pi} [\operatorname{Re} F(\theta) \cos n\theta - \operatorname{Im} F(\theta) \sin n\theta]\, d\theta$$
$$+ i \int_{-\pi}^{\pi} [\operatorname{Re} F(\theta) \sin n\theta + \operatorname{Im} F(\theta) \cos n\theta]\, d\theta.$$

However, by the given conditions on $F(\theta)$, the second integrand is an odd function of θ and the second integral is zero. Hence a_n is real.

THEOREM. *If $F(z) \in H^p$, has real coefficients and satisfies*

$$\lambda \operatorname{Re} F(\theta) = -\operatorname{Im} F(\theta)$$

a.e. in $(0, \pi)$, *then* $\operatorname{Re} F(\theta) \in L^p[0, \pi]$ *and*

$$\int_0^{\pi} \operatorname{Re} F(\theta) (\cos n\theta + \lambda \sin n\theta)\, d\theta = 0 \qquad n = 1, 2, 3, \ldots .$$

PROOF. It is obvious that $\operatorname{Re} F(\theta) \in L^p[0, \pi]$ since $F(\theta) \in L^p[-\pi, \pi]$. Since $F(z) \in H^p$ and has real coefficients the lemma applies i.e. $\operatorname{Re} F(\theta)$ is even and $\operatorname{Im} F(\theta)$ is odd. As $F(z) \in H^p$

$$\int_{-\pi}^{\pi} F(\theta)\, e^{in\theta}\, d\theta = 0 \qquad n = 1, 2, 3, \ldots .$$

Therefore

$$\int_{-\pi}^{\pi} [\operatorname{Re} F(\theta) \cos n\theta - \operatorname{Im} F(\theta) \sin n\theta]\, d\theta$$
$$+ i \int_{-\pi}^{\pi} [\operatorname{Re} F(\theta) \sin n\theta + \operatorname{Im} F(\theta) \cos n\theta]\, d\theta$$
$$= 0 \qquad n = 1, 2, 3, \ldots .$$

The second integrand is odd and hence the second integral is zero. We therefore have:

$$\int_{-\pi}^{\pi} [\operatorname{Re} F(\theta) \cos n\theta - \operatorname{Im} F(\theta) \sin n\theta]\, d\theta = 0 \qquad n = 1, 2, \ldots.$$

Since the integrand is even, we can reduce the interval of integration to $[0, \pi]$ and by substituting $\lambda \operatorname{Re} F(\theta)$ for $-\operatorname{Im} F(\theta)$ we get

$$\int_{0}^{\pi} [\operatorname{Re} F(\theta) (\cos n\theta + \lambda \sin n\theta)]\, d\theta = 0 \qquad n = 1, 2, 3, \ldots.$$

In a previous paper ([1], p. 306) we proved the following: If $g(\theta) \in L^p[0, \pi]$ and

$$\int_{0}^{\pi} g(\theta) [\cos n\theta + \lambda \sin n\theta]\, d\theta = 0 \qquad n = 1, 2, 3, \ldots$$

then $g(\theta) = C \tan^{\beta}(\theta/2)$ where C is some constant (and $C = 0$ if $p \mid \beta \mid \geqslant 1$).

Note. We can always assume that C is real.

Combining this fact with the previous theorem, we have the following

THEOREM. *If $F(z) \in H^p$, has real coefficients and satisfies*

$$\lambda \operatorname{Re} F(\theta) = -\operatorname{Im} F(\theta)$$

a.e. in $[0, \pi]$, *then* $\operatorname{Re} F(\theta) = C \tan^{\beta}(\theta/2)$.

If $p \mid \beta \mid \geqslant 1$ then C must be 0. That forces $\operatorname{Re} F(\theta)$ to be zero a.e. in $[0, \pi]$ and together with $\lambda \operatorname{Re} F(\theta) = -\operatorname{Im} F(\theta)$ we get $\operatorname{Im} F(\theta) = 0$ a.e. in $[0, \pi]$. By the evenness of $\operatorname{Re} F(\theta)$ and the oddness of $\operatorname{Im} F(\theta)$ we have $F(\theta) = 0$ a.e. in $[-\pi, \pi]$. That in turn forces $F(z)$ to be identically zero.

On the other hand, if $p \mid \beta \mid < 1$, let

$$G(z) = \sqrt{1 + \lambda^2}(1 - z/1 + z)^{\beta}.$$

Then

$$G(z) \in H^p, \text{ and } G(\theta) = G(e^{i\theta}) = \sqrt{1 + \lambda^2}\,(-i \tan \theta/2)^{\beta}.$$

Take any $\theta \in (0, \pi)$. Then $G(\theta) = (1 - i\lambda) \tan^{\beta} (\theta/2)$. Therefore,

$$\lambda \operatorname{Re} G(\theta) = -\operatorname{Im} G(\theta) = \lambda \tan^{\beta} (\theta/2)$$

a.e. in $(0, \pi)$. It is easily shown that

$$\operatorname{Re} G(\theta) = \operatorname{Re} G(-\theta) \qquad \text{and} \qquad \operatorname{Im} G(\theta) = -\operatorname{Im} G(-\theta)$$

a.e. in $(0, \pi)$. Therefore assuming the existence of a function $F(z) \in H^p$ satisfying our boundary conditions, we would have $F(\theta) = CG(\theta)$ a.e. in $(-\pi, \pi)$. That of course implies that $F(z)$ is identical with $CG(z)$ or that $F(z) = C_1(1 - z/1 + z)^{\beta}$ where C_1 is a real constant. We have therefore proven our main theorem.

REFERENCES

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